

PROJECTIVE NESTED CARTESIAN CODES

CÍCERO CARVALHO, V. G. LOPEZ NEUMANN, AND HIRAM H. LÓPEZ

ABSTRACT. In this paper we introduce a new type of code, called projective nested cartesian code. It is obtained by the evaluation of homogeneous polynomials of a fixed degree on a certain subset of $\mathbb{P}^n(\mathbb{F}_q)$, and they may be seen as a generalization of the so-called projective Reed-Muller codes. We calculate the length and the dimension of such codes, a lower bound for the minimum distance and the exact minimum distance in a special case (which includes the projective Reed-Muller codes). At the end we show some relations between the parameters of these codes and those of the affine cartesian codes.

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1. INTRODUCTION

Let $K := \mathbb{F}_q$ be a field with q elements and let A_0, \dots, A_n be a collection of non-empty subsets of K . Consider a *projective cartesian set*

$$\mathcal{X} := [A_0 \times A_1 \times \dots \times A_n] := \{(a_0 : \dots : a_n) \mid a_i \in A_i \text{ for all } i\} \subset \mathbb{P}^n,$$

where \mathbb{P}^n is a projective space over the field K .

In what follows d_i denotes $|A_i|$, the cardinality of A_i for $i = 0, \dots, n$. We shall always assume that $2 \leq d_i \leq d_{i+1}$ for all i . The case $d_0 = d_1 = \dots = d_l = 1$, for some l , is treated separately (Lemma 2.5).

Let $S := K[X_0, \dots, X_n]$ be a polynomial ring over the field K , let P_1, \dots, P_m be the points of \mathcal{X} written with the usual (see e.g. [10], [7], [3]) representation for projective points, that is, zeros to the left and the first nonzero entry equal 1, and let S_d be the K -vector space of all homogeneous polynomials of S of degree d together with the zero polynomial. The *evaluation map*

$$\varphi_d: S_d \longrightarrow K^{|\mathcal{X}|}, \quad f \mapsto (f(P_1), \dots, f(P_m)),$$

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defines a linear map of K -vector spaces. The image of φ_d , denoted by $C_{\mathcal{X}}(d)$, defines a linear code (as usual by a *linear code* we mean a linear subspace of $K^{|\mathcal{X}|}$). We call $C_{\mathcal{X}}(d)$ a *projective cartesian code* of order d defined over A_0, \dots, A_n . An important special case of such codes, which served as motivation for our work, happens when $A_i = K$ for all $i = 0, \dots, n$. Then we have $\mathcal{X} = \mathbb{P}^n$ and $C_{\mathcal{X}}(d)$ is the so-called projective Reed-Muller code (of order d), as defined and studied in [7] or [10], for example.

The *dimension* and the *length* of $C_{\mathcal{X}}(d)$ are given by $\dim_K C_{\mathcal{X}}(d)$ (dimension as K -vector space) and $|\mathcal{X}|$ respectively. The *minimum distance* of $C_{\mathcal{X}}(d)$ is given by

$$\delta_{\mathcal{X}}(d) = \min\{\|\varphi_d(f)\| : \varphi_d(f) \neq 0; f \in S_d\},$$

where $\|\varphi_d(f)\|$ is the number of non-zero entries of $\varphi_d(f)$. These are the main parameters of the code $C_{\mathcal{X}}(d)$ and they are presented in the main results of this paper, although we find the minimum distance only when the A_i 's satisfy certain conditions (Definition 2.1).

In the next section we compute the length and the dimension of $C_{\mathcal{X}}(d)$, and to do this we use some concepts of commutative algebra which we now recall. The *vanishing ideal* of $\mathcal{X} \subset \mathbb{P}^n$, denoted by $I(\mathcal{X})$, is the ideal of S generated by the homogeneous polynomials that vanish on all points of \mathcal{X} . We are interested in the algebraic invariants (degree, Hilbert function) of $I(\mathcal{X})$, because the kernel of the evaluation map, φ_d , is precisely $I(\mathcal{X})_d$, where $I(\mathcal{X})_d := S_d \cap I(\mathcal{X})$. In general, for any subset (ideal or not) \mathcal{F} of S we define $\mathcal{F}_d := \mathcal{F} \cap S_d$. The *Hilbert function* of $S/I(\mathcal{X})$ is given by

$$H_{\mathcal{X}}(d) := \dim_K(S_d/I(\mathcal{X})_d),$$

so $H_{\mathcal{X}}(d)$ is precisely the dimension of $C_{\mathcal{X}}(d)$. According to [6, Lecture 13], we have that $H_{\mathcal{X}}(d) = |\mathcal{X}|$ for $d \geq |\mathcal{X}| - 1$, which means that the length $|\mathcal{X}|$ of $C_{\mathcal{X}}(d)$ is the *degree* of $S/I(\mathcal{X})$ in the sense of algebraic geometry [6, p. 166].

In section 3 we determine the minimum distance of a particular type of projective cartesian code which is defined by product of subfields of K (see Definition 3.4). We will use more than once results about affine cartesian codes, which we now recall.

Let A_1, \dots, A_n be, as above, a collection of non-empty subsets of K , write d_i for the cardinality of A_i , $i = 1, \dots, n$, and set $\mathcal{Y} := A_1 \times \dots \times A_n \subset \mathbb{A}^n$, where \mathbb{A}^n is the n -dimensional affine space defined over K . For a nonnegative integer d write $S_{\leq d}$ for the K -linear subspace of K^n formed by the polynomials in $K[X]$ of degree up to d together with the zero polynomial. Clearly $|\mathcal{Y}| = \prod_{i=1}^n d_i =: \tilde{m}$ and let $Q_1, \dots, Q_{\tilde{m}}$ be the points of \mathcal{Y} . Define $\phi_d : S_{\leq d} \rightarrow K^{\tilde{m}}$ as the evaluation morphism $\phi_d(g) = (g(Q)_1, \dots, g(Q_{\tilde{m}}))$.

Definition 1.1. The image $C_{\mathcal{Y}}^*(d)$ of ϕ_d is a subvector space of $K^{\tilde{m}}$ called the *affine cartesian code* (of order d) defined over the sets A_1, \dots, A_n .

These codes were introduced in [8], and also appeared independently and in a generalized form in [5]. They are a type of affine variety code, as defined in [4]. In [8] the authors prove that we may ignore sets with just one element, and moreover may always assume that $2 \leq d_1 \leq \dots \leq d_n$. They also completely determine the parameters of these codes, which are as follows.

Theorem 1.2. [8, Thm. 3.1 and Thm. 3.8]

1) The dimension of $C_{\mathcal{Y}}^*(d)$ is \tilde{m} (i.e. ϕ_d is surjective) if $d \geq \sum_{i=1}^n (d_i - 1)$, and for $0 \leq d < \sum_{i=1}^n (d_i - 1)$ we have

$$\dim(C_{\mathcal{Y}}^*(d)) = \binom{n+d}{d} - \sum_{i=1}^n \binom{n+d-d_i}{d-d_i} + \dots + (-1)^j \sum_{1 \leq i_1 < \dots < i_j \leq n} \binom{n+d-d_{i_1}-\dots-d_{i_j}}{d-d_{i_1}-\dots-d_{i_j}} + \dots + (-1)^n \binom{n+d-d_1-\dots-d_n}{d-d_1-\dots-d_n}$$

where we set $\binom{a}{b} = 0$ if $b < 0$.

2) The minimum distance $\delta_{\mathcal{Y}}^*(d)$ of $C_{\mathcal{Y}}^*(d)$ is 1, if $d \geq \sum_{i=1}^n (d_i - 1)$, and for $0 \leq d < \sum_{i=1}^n (d_i - 1)$ we have

$$\delta_{\mathcal{Y}}^*(d) = (d_{k+1} - \ell) \prod_{i=k+2}^n d_i$$

where k and ℓ are uniquely defined by $d = \sum_{i=1}^k (d_i - 1) + \ell$ with $0 \leq \ell < d_{k+1} - 1$ (if $k+1 = n$ we understand that $\prod_{i=k+2}^n d_i = 1$, and if $d < d_1 - 1$ then we set $k = 0$ and $\ell = d$).

We will also use a result from [8] in which the authors determine the (homogeneous) ideal of the set $\bar{\mathcal{Y}} := [1 \times A_1 \times \dots \times A_n]$ (in what follows we use, in a cartesian product, 1 to denote the set $\{1\}$ and 0 to denote the set $\{0\}$). Observe that this set may be viewed as the closure of \mathcal{Y} in \mathbb{P}^n .

Theorem 1.3. [8, Thm. 2.5]

$$I(\bar{\mathcal{Y}}) = \langle \Pi_{a_1 \in A_1} (X_1 - a_1 X_0), \dots, \Pi_{a_n \in A_n} (X_n - a_n X_0) \rangle$$

In [1] there are results on higher Hamming weights of affine cartesian codes, and also a proof of the minimum distance formula stated above which is simpler from the one found

in [8] and uses methods similar to the ones used here. We will need a result from [1] which we reproduce here for the reader's convenience.

Lemma 1.4. [1, Lemma 2.1] *Let $0 < d_1 \leq \dots \leq d_n$ and $0 \leq s \leq \sum_{i=1}^n (d_i - 1)$ be integers. Let $m(a_1, \dots, a_n) = \prod_{i=1}^n (d_i - a_i)$, where $0 \leq a_i < d_i$ is an integer for all $i = 1, \dots, n$. Then*

$$\min\{m(a_1, \dots, a_n) \mid a_1 + \dots + a_n \leq s\} = (d_{k+1} - \ell) \prod_{i=k+2}^n d_i$$

where k and ℓ are uniquely defined by $s = \sum_{i=1}^k (d_i - 1) + \ell$, with $0 \leq \ell < d_{k+1} - 1$ (if $s < d_1 - 1$ then take $k = 0$ and $\ell = s$, if $k+1 = n$ then we understand that $\prod_{i=k+2}^n d_i = 1$).

2. LENGTH AND DIMENSION

In this section we define the projective nested cartesian codes and compute their length and dimension. We keep the notation and definitions used in Section 1.

For A, B subsets of K we write $A^{\neq 0}$ to denote the set $A \setminus \{0\}$ and we define $\frac{A}{B} := \{\frac{a}{b} \mid a \in A, b \in B^{\neq 0}\}$.

Definition 2.1. Let A_0, A_1, \dots, A_n be a collection of non-empty subsets of K such that

- (i) for all $i = 0, \dots, n$ we have $0 \in A_i$,
- (ii) for every $i = 1, \dots, n$ we have $\frac{A_j}{A_{i-1}} \subset A_j$ for $j = i, \dots, n$.

Under these conditions, the projective cartesian set $\mathcal{X} = [A_0 \times A_1 \times \dots \times A_n]$ is called *projective nested cartesian set*. For any $d \geq 0$ the associated linear code $C_{\mathcal{X}}(d)$ is called a *projective nested cartesian code*.

Remark 2.2. For A, B subsets of K with $0 \in A \cap B$ we have $\frac{A}{B} \subset A \iff AB \subset A$. This is because if $\frac{A}{B} \subset A$ then $b \in B^{\neq 0}$ defines a bijection $A \rightarrow A$ given by $a \mapsto a/b$, with inverse $a \mapsto ab$. In particular condition (ii) of Definition 2.1 is equivalent to the following condition:

- (ii') for every $i = 1, \dots, n$ we have $A_j A_{i-1} \subset A_j$ for $j = i, \dots, n$.

Remark 2.3. If we take $A_i = K$ for all $i = 0, \dots, n$ then the conditions of Definition 2.1 are satisfied, so \mathbb{P}^n is a projective nested cartesian set and the projective Reed-Muller codes are projective nested cartesian codes.

Lemma 2.4. *If $\mathcal{X} = [A_0 \times A_1 \times A_2 \times \cdots \times A_n]$ is a projective nested cartesian set then*

$$I(\mathcal{X}) = \left\langle X_i \prod_{a_j \in A_j} (X_j - a_j X_i), i < j; i, j = 0, \dots, n \right\rangle.$$

Proof. We will make an induction on n . If $n = 1$ then $\mathcal{X} = [1 \times A_n] \cup \{(0 : 1)\}$ and from Theorem 1.3 we get $I(\mathcal{X}) = \langle X_0 \prod_{a_1 \in A_1} (X_1 - a_1 X_0) \rangle$. Now we assume that the result is valid for $n - 1$. Take $C_1 := [1 \times A_1 \times A_2 \times \cdots \times A_n]$, $C_0 := [A_1 \times A_2 \times \cdots \times A_n]$ and $F \in I(\mathcal{X})$. Let m be an element of C_0 and write

$$F = F_1 X_0 + F_2,$$

where $F_2 \in K[X_1, \dots, X_n]$. As \mathcal{X} is a projective nested cartesian set, $\mathcal{X} = C_1 \cup [0 \times C_0]$, so $[1, m], [0, m] \in \mathcal{X}$. We have $0 = F(0, m) = F_2(m)$, then $F_2 \in I(C_0)$ and by induction

$$F_2 \in \left\langle X_i \prod_{a_j \in A_j} (X_j - a_j X_i), i < j; i, j = 1, \dots, n \right\rangle.$$

We know also $0 = F(1, m) = F_1(m)$, then $F_1 \in I(C_1)$ and from Theorem 1.3 we get

$$F_1 \in \left\langle \prod_{a_i \in A_i} (X_i - a_i X_0), i = 1, \dots, n \right\rangle.$$

As $F = F_1 X_0 + F_2$ the result is true. \square

Observe that if $d_0 = \cdots = d_n = 1$ then $\mathcal{X} = \emptyset$ because $A_0 = \cdots = A_n = 0$. Thus we must have $d_i > 1$ for some $i \in \{0, \dots, n\}$ and the next result proves that we may disregard the sets A_i such that $d_i = 1$.

Lemma 2.5. *Let $\mathcal{X} = [A_0 \times \cdots \times A_n]$ be a projective nested cartesian set. If $d_0 = \cdots = d_l = 1 < d_{l+1}$, with $0 \leq l \leq n - 1$, then $C_{\mathcal{X}}(d) = C_{\mathcal{X}'}(d)$, where $\mathcal{X}' = [A_{l+1} \times \cdots \times A_n]$.*

Proof. The condition $d_0 = \cdots = d_l = 1$ means $A_0 = \cdots = A_l = \{0\}$, so we write a point of \mathcal{X} as $(\mathbf{0}, Q)$, where $Q \in \mathcal{X}'$. To prove that $C_{\mathcal{X}}(d) \subset C_{\mathcal{X}'}(d)$, let $u = (f(\mathbf{0}, Q_1), \dots, f(\mathbf{0}, Q_m))$ be an element of $C_{\mathcal{X}}(d)$. Then $f \in S_d$ and $f = X_0 f_0 + \cdots + X_l f_l + F$, with $F \in K[X_{l+1}, \dots, X_n]_d$. Clearly $f(\mathbf{0}, Q_i) = F(Q_i)$ for all $i = 1, \dots, m$ so $u = (F(Q_1), \dots, F(Q_m)) \in C_{\mathcal{X}'}(d)$. Conversely let $v = (g(Q_1), \dots, g(Q_m)) \in C_{\mathcal{X}'}(d)$, where $g \in K[X_{l+1}, \dots, X_n]_d \subset S_d$, then $u = (g(\mathbf{0}, Q_1), \dots, g(\mathbf{0}, Q_m)) \in C_{\mathcal{X}}(d)$. \square

Definition 2.6. Let $\mathcal{X} = [A_0 \times \cdots \times A_n]$ be a projective nested cartesian set. To compute the Hilbert function of $I(\mathcal{X})$ we define

$$\begin{aligned}\mathcal{X}_i &:= [A_{n-i} \times \cdots \times A_n], \text{ and } I(\mathcal{X}_i) \subset K[X_{n-i}, \dots, X_n], \text{ for } i = 0, \dots, n; \\ \mathcal{X}_i^* &:= [1 \times A_{n+1-i} \times \cdots \times A_n], \text{ and } I(\mathcal{X}_i^*) \subset K[X_{n-i}, \dots, X_n], \text{ for } i = 1, \dots, n.\end{aligned}$$

Lemma 2.7. For any positive integer d , $H_{\mathcal{X}_n}(d) = H_{\mathcal{X}_{n-1}}(d) + H_{\mathcal{X}_n^*}(d-1)$.

Proof. We know that $S_d = K[X_1, \dots, X_n]_d \oplus X_0 K[X_0, \dots, X_n]_{d-1}$. Let $f \in I_{\mathcal{X}_n}(d)$, then $f = h + X_0 g$, where $h \in K[X_1, \dots, X_n]_d$ and $g \in K[X_0, \dots, X_n]_{d-1}$. By definition 2.6, it is easy to see that $h \in I_{\mathcal{X}_{n-1}}(d)$ and $g \in I_{\mathcal{X}_n^*}(d-1)$ and conversely, if $h \in I_{\mathcal{X}_{n-1}}(d)$ and $g \in I_{\mathcal{X}_n^*}(d-1)$, then $h + X_0 g \in I_{\mathcal{X}_n}(d)$. Thus $I_{\mathcal{X}_n}(d) = I_{\mathcal{X}_{n-1}}(d) \oplus X_0 I_{\mathcal{X}_n^*}(d-1)$. Then

$$\begin{aligned}S_d / I_{\mathcal{X}_n}(d) &\simeq K[X_1, \dots, X_n]_d / I_{\mathcal{X}_{n-1}}(d) \oplus X_0 K[X_0, \dots, X_n]_{d-1} / X_0 I_{\mathcal{X}_n^*}(d-1) \\ &\simeq K[X_1, \dots, X_n]_d / I_{\mathcal{X}_{n-1}}(d) \oplus K[X_0, \dots, X_n]_{d-1} / I_{\mathcal{X}_n^*}(d-1)\end{aligned}$$

which completes the proof. \square

Lemma 2.8. Let $\mathcal{X} = [A_0 \times \cdots \times A_n]$ be a projective nested cartesian set. The Hilbert function of $S/I(\mathcal{X})$ is given by

$$\begin{aligned}H_{\mathcal{X}}(d) &= 1 + \sum_{j=1}^n \left[\binom{j+d-1}{d-1} - \sum_{i=n+1-j}^n \binom{j+d-1-d_i}{d-1-d_i} + \cdots + \right. \\ &\quad (-1)^k \sum_{n+1-j \leq i_1 < \cdots < i_k \leq n} \binom{j+d-1-(d_{i_1} + \cdots + d_{i_k})}{d-1-(d_{i_1} + \cdots + d_{i_k})} + \cdots + \\ &\quad \left. (-1)^j \binom{j+d-1-(d_{n+1-j} + \cdots + d_n)}{d-1-(d_{n+1-j} + \cdots + d_n)} \right].\end{aligned}$$

Proof. Using Lemma 2.7 we have $H_{\mathcal{X}}(d) = H_{\mathcal{X}_0}(d) + \sum_{j=1}^n H_{\mathcal{X}_j^*}(d-1)$. As $\mathcal{X}_0 = [1]$, then

$I(\mathcal{X}_0) = 0$ and $H_{\mathcal{X}_0} = 1$. From Theorem 1.2 (1) we get

$$\begin{aligned}H_{\mathcal{X}_j^*}(d-1) &= \binom{j+d-1}{d-1} - \sum_{i=n+1-j}^n \binom{j+d-1-d_i}{d-1-d_i} + \cdots + \\ &\quad (-1)^k \sum_{n+1-j \leq i_1 < \cdots < i_k \leq n} \binom{j+d-1-(d_{i_1} + \cdots + d_{i_k})}{d-1-(d_{i_1} + \cdots + d_{i_k})} + \cdots + \\ &\quad (-1)^j \binom{j+d-1-(d_{n+1-j} + \cdots + d_n)}{d-1-(d_{n+1-j} + \cdots + d_n)}.\end{aligned}$$

We come to the main result of this section.

Theorem 2.9. *Let $C_{\mathcal{X}}(d)$ be a projective nested cartesian code over A_0, \dots, A_n . The length of the code is given by $m = 1 + \sum_{i=1}^n d_i \cdots d_n$ and its dimension by*

$$\begin{aligned} \dim_K C_{\mathcal{X}}(d) = & 1 + \sum_{j=1}^n \left[\binom{j+d-1}{d-1} - \sum_{i=n+1-j}^n \binom{j+d-1-d_i}{d-1-d_i} + \cdots + \right. \\ & (-1)^k \sum_{n+1-j \leq i_1 < \cdots < i_k \leq n} \binom{j+d-1-(d_{i_1}+\cdots+d_{i_k})}{d-1-(d_{i_1}+\cdots+d_{i_k})} + \cdots + \\ & \left. (-1)^j \binom{j+d-1-(d_{n+1-j}+\cdots+d_n)}{d-1-(d_{n+1-j}+\cdots+d_n)} \right]. \end{aligned}$$

Proof. As $\mathcal{X} = [A_0 \times A_1 \times \cdots \times A_n]$ is a projective nested cartesian set, then

$$\begin{aligned} \mathcal{X} = & \left[A_0^{\neq 0} \times A_1 \times A_2 \times \cdots \times A_n \right] \cup \\ & \left[0 \times A_1^{\neq 0} \times A_2 \times \cdots \times A_n \right] \cup \\ & \vdots \\ & \left[0 \times 0 \times 0 \times \cdots \times A_{n-1}^{\neq 0} \times A_n \right] \cup \\ & [0 \times 0 \times 0 \times \cdots \times 0 \times 1]. \end{aligned}$$

Condition (ii) of Definition 2.1 allows us change $A_i^{\neq 0}$ for 1 for all $i = 0, \dots, n-1$ so we get $|\mathcal{X}| = 1 + \sum_{i=1}^n d_i \cdots d_n$. As the kernel of the evaluation map φ_d is $S_d \cap I(\mathcal{X})$, the Hilbert function of $S/I(\mathcal{X})$ agrees with the dimension of $C_{\mathcal{X}}(d)$, so, by Lemma 2.8 we have the dimension. □

To finish this section, we show that for the graded lexicographic monomial order \prec in S , where $X_0 \prec \cdots \prec X_n$, the set

$$\mathcal{G} := \left\{ X_i \prod_{a_j \in A_j} (X_j - a_j X_i), i < j; i, j = 0, \dots, n \right\}$$

is a Gröbner basis of the ideal $I(\mathcal{X})$ with respect to a graded monomial order. In what follows, M denotes a monomial in S .

Definition 2.10. The *footprint* (with respect to a monomial order \prec) of an ideal $I \subset S$, denoted by $\Delta(I)$, is the set of monomials which are not leading monomials of any polynomial in I . If $G = \{g_1, g_2, \dots, g_s\}$ is a subset of $K[X]$, we set $\Delta(G) := \{M \mid \text{for all } i, \text{lm}(g_i) \nmid M\}$, where $\text{lm}(g)$ denotes the leading monomial of $g \in S$. We write $\Delta(G)_d$ to denote the set of monomials in $\Delta(G)$ of degree equal to d , for any integer $d \geq 0$.

Lemma 2.11. Fix a graded monomial order in S . Let I be a homogeneous ideal of S and $G = \{g_1, g_2, \dots, g_s\}$ a set of generators of I . The set G is a Gröbner Basis of I if and only if the Hilbert function of I is given by $H_I(d) = \#\Delta(G)_d$, for all $d \geq 0$.

Proof. We know that $\langle \text{lm}(g_1), \dots, \text{lm}(g_s) \rangle \subset \langle \text{lm}(I) \rangle$, where equality holds if and only if G is a Gröbner basis. This means that $\Delta(I) \subset \Delta(G)$ and equality holds if G is a Gröbner basis. As the number of elements of $\Delta(I)_d$ is equal to $H_I(d)$, we have our result. \square

From now on we choose the graded lexicographic monomial order \prec in S , where $X_0 \prec \dots \prec X_n$.

Lemma 2.12. The number of elements of $\Delta(\mathcal{G})_d$ is given by

$$\begin{aligned} & \binom{n+d}{n} - \sum_{j=1}^n \left(\binom{n+d-d_j}{n} - \binom{n-j+d-d_j}{n-j} \right) + \dots + \\ & + (-1)^k \sum_{1 \leq j_1 < j_2 < \dots < j_k \leq n} \left(\binom{n+d-(d_{j_1}+\dots+d_{j_k})}{n} - \binom{n-j_1+d-(d_{j_1}+\dots+d_{j_k})}{n-j_1} \right) + \\ & + \dots + (-1)^n \binom{n+d-(d_1+\dots+d_n+1)}{n}. \end{aligned}$$

Proof. Observe that $\Delta(\mathcal{G}) = \{M \mid X_i X_j^{d_j} \nmid M, 0 \leq i < j \leq n\}$. For $1 \leq j \leq n$, we define $\mathcal{M}_j := \{M \mid \text{there is } i, 0 \leq i < j, X_i X_j^{d_j} \mid M\}$. Then $\Delta(\mathcal{G}) = \mathcal{M}_S - \left(\bigcup_{j=1}^n \mathcal{M}_j \right)$, where \mathcal{M}_S is the set of all monomials in S . Therefore, when we count the number of monomials of degree d in $\Delta(\mathcal{G})$, from the inclusion-exclusion theorem we get

$$\begin{aligned} \Delta(\mathcal{G})_d &= \#(\mathcal{M}_S)_d - \sum_{j=1}^n \#(\mathcal{M}_j)_d + \sum_{j_1 < j_2} \#(\mathcal{M}_{j_1} \cap \mathcal{M}_{j_2})_d - \dots \\ &+ (-1)^k \sum_{j_1 < j_2 < \dots < j_k} \#(\mathcal{M}_{j_1} \cap \mathcal{M}_{j_2} \cap \dots \cap \mathcal{M}_{j_k})_d + \dots \\ &+ (-1)^n \#(\mathcal{M}_1 \cap \mathcal{M}_2 \cap \dots \cap \mathcal{M}_n)_d. \end{aligned}$$

Clearly $\#(\mathcal{M}_S)_d = \binom{n+d}{n}$. Let $j \in \{1, \dots, n\}$ and let $M = X_0^{\alpha_0} \cdots X_n^{\alpha_n} \in (\mathcal{M}_j)_d$, then there exists $i < j$, such that $\alpha_i \geq 1$ and $\alpha_j \geq d_j$. Taking $\beta_j = \alpha_j - d_j$ and for $k \neq j$, $\beta_k = \alpha_k$, we have that $\#(\mathcal{M}_j)_d$ is the number of solutions of $\beta_0 + \cdots + \beta_n = d - d_j$, such that $\beta_0 + \cdots + \beta_{j-1} \geq 1$. Then $\#(\mathcal{M}_j)_d$ is the number of solutions of $\beta_0 + \cdots + \beta_n = d - d_j$ minus the number of solutions of $\beta_j + \cdots + \beta_n = d - d_j$. This means

$$\#(\mathcal{M}_j)_d = \binom{n+d-d_j}{n} - \binom{n-j+d-d_j}{n-j}.$$

Now let $M = X_0^{\alpha_0} \cdots X_n^{\alpha_n} \in (\mathcal{M}_{j_1} \cap \cdots \cap \mathcal{M}_{j_k})_d$, then there exists $i < j_1$, such that $\alpha_i \geq 1$ and $\alpha_{j_w} \geq d_{j_w}$, for $1 \leq w \leq k$. Taking $\beta_{j_w} = \alpha_{j_w} - d_{j_w}$, for $1 \leq w \leq k$, with $l \neq j_w$ and $\beta_l = \alpha_l$, we get that $\#(\mathcal{M}_{j_1} \cap \cdots \cap \mathcal{M}_{j_k})_d$ is the number of solutions of $\beta_0 + \cdots + \beta_n = d - (d_{j_1} + \cdots + d_{j_k})$ minus the number of solutions of $\beta_{j_1} + \cdots + \beta_n = d - (d_{j_1} + \cdots + d_{j_k})$, hence

$$\#(\mathcal{M}_{j_1} \cap \cdots \cap \mathcal{M}_{j_k})_d = \binom{n+d-(d_{j_1} + \cdots + d_{j_k})}{n} - \binom{n-j_1+d-(d_{j_1} + \cdots + d_{j_k})}{n-j_1}.$$

For $k = n$ we have $\binom{n+d-(d_1+\cdots+d_n)}{n} - \binom{n-1+d-(d_1+\cdots+d_n)}{n-1} = \binom{n+d-(d_1+\cdots+d_{n+1})}{n}$. \square

We use the next well-known combinatorial result to check that $H_{\mathcal{X}}(d) = \#\Delta(\mathcal{G})_d$ for all $d \geq 0$.

Lemma 2.13. *Let a, b be non-negative integers. Then $\sum_{j=0}^a \binom{j+b-1}{j} = \binom{a+b}{a}$.*

Proposition 2.14. *Let $\mathcal{X} = [A_0 \times \cdots \times A_n]$ be a projective nested cartesian set. The set $\mathcal{G} = \left\{ X_i \prod_{a_j \in A_j} (X_j - a_j X_i), i < j; i, j = 0, \dots, n \right\}$ is a Gröbner basis for $I(\mathcal{X})$.*

Proof. From Lemma 2.11 we only need to compare the formulas of Lemmas 2.8 and 2.12. On the formula for the Hilbert Function, we distribute the sum, use Lemma 2.13 and compare term by term with the formula for the footprint. The first term is

$$1 + \sum_{j=1}^n \binom{j+d-1}{d-1} = \sum_{j=0}^n \binom{j+d-1}{j} = \binom{n+d}{n},$$

the second term is

$$\begin{aligned}
\sum_{j=1}^n \sum_{i=n+1-j}^n \binom{j+d-1-d_i}{d-1-d_i} &= \sum_{i=1}^n \sum_{j=n+1-i}^n \binom{j+d-1-d_i}{j} \\
&= \sum_{j=1}^n \sum_{i=n+1-j}^n \binom{i+d-1-d_j}{i} \\
&= \sum_{j=1}^n \left(\sum_{i=0}^n \binom{i+d-d_j-1}{i} - \sum_{i=0}^{n-j} \binom{i+d-d_j-1}{i} \right) \\
&= \sum_{j=1}^n \left(\binom{n+d-d_j}{n} - \binom{n-j+d-d_j}{n-j} \right),
\end{aligned}$$

and the general term is

$$\begin{aligned}
\sum_{j=1}^n \sum_{n+1-j \leq i_1 < \dots < i_k \leq n} \binom{j+d-1-(d_{i_1}+\dots+d_{i_k})}{d-1-(d_{i_1}+\dots+d_{i_k})} &= \\
\sum_{1 \leq i_1 < \dots < i_k \leq n} \sum_{j=n+1-i_1}^n \binom{j+d-1-(d_{i_1}+\dots+d_{i_k})}{j} &= \\
\sum_{1 \leq i_1 < \dots < i_k \leq n} \left(\binom{n+d-(d_{i_1}+\dots+d_{i_k})}{n} - \binom{n-i_1+d-(d_{i_1}+\dots+d_{i_k})}{n-i_1} \right). &
\end{aligned}$$

Finally, for the last term, the sum on the formula for the Hilbert function has only one term, and $\binom{n+d-1-(d_1+\dots+d_n)}{d-1-(d_1+\dots+d_n)} = \binom{n+d-(d_1+\dots+d_n+1)}{n}$, which proves the Proposition. \square

3. MINIMUM DISTANCE

We start this section by presenting an upper bound for the minimum distance of projective nested cartesian codes. Instead of $f(X_0, \dots, X_n)$ we write simply $f(X)$ for a polynomial in S .

Lemma 3.1. *If \mathcal{X} is the projective nested cartesian set over A_0, \dots, A_n , then the minimum distance of $C_{\mathcal{X}}(d)$ satisfies $\delta_{\mathcal{X}}(d) \leq (d_{k+1} - \ell) d_{k+2} \cdots d_n$ if $1 \leq d \leq \sum_{i=1}^n (d_i - 1)$, and $\delta_{\mathcal{X}}(d) = 1$ in otherwise, where $0 \leq k \leq n-1$ and $0 \leq \ell < d_{k+1} - 1$ are the unique integers such that $d - 1 = \sum_{i=1}^k (d_i - 1) + \ell$.*

Proof. For all $i = 0, \dots, n$ choose $a_i \in A_i$. It is easy to see that $f(X) = X_0 \prod_{i=1}^n \prod_{a \in A_i}^{a \neq a_i} (X_i - aX_0)$ of degree $\sum_{i=1}^n (d_i - 1) + 1$ is zero for all points of \mathcal{X} except $(1 : a_1 : \dots : a_n)$. Thus for $d > \sum_{i=1}^n (d_i - 1)$ we get $\delta_{\mathcal{X}}(d) = 1$. Let $B_{k+1} \subset A_{k+1}$ be a set with ℓ elements. For $d - 1 = \sum_{i=1}^k (d_i - 1) + \ell$, taking $f(X) = X_0 \left(\prod_{i=1}^k \prod_{a \in A_i}^{a \neq a_i} (X_i - aX_0) \right) \left(\prod_{a \in B_{k+1}} (X_{k+1} - aX_0) \right)$, we obtain the desired inequality. \square

We believe that this upper bound is actually the true value of the minimum distance.

Conjecture 3.2. *If \mathcal{X} is the projective nested cartesian set over A_0, \dots, A_n , then the minimum distance of $C_{\mathcal{X}}(d)$ is given by*

$$\delta_{\mathcal{X}}(d) = \begin{cases} (d_{k+1} - \ell) d_{k+2} \cdots d_n & \text{if } 1 \leq d \leq \sum_{i=1}^n (d_i - 1), \\ 1 & \text{if } \sum_{i=1}^n (d_i - 1) < d, \end{cases}$$

where $0 \leq k \leq n - 1$ and $0 \leq \ell < d_{k+1} - 1$ are the unique integers such that $d - 1 = \sum_{i=1}^k (d_i - 1) + \ell$.

We will prove this conjecture in the special case where the sets A_i are subfields of K (so it includes the projective Reed-Muller codes). Before we define this class of codes we prove an auxiliary result.

Lemma 3.3. *Let $\mathcal{X} = [A_0 \times \dots \times A_n]$ be a projective nested cartesian set. For all $j = 0, \dots, n$ let $a_j \in A_j^{\neq 0}$ and define $B_j = a_j^{-1} A_j$. Then $\mathcal{Y} = [B_0 \times \dots \times B_n]$ is a projective nested cartesian set such that $1 \in B_j$, for all $j = 0, \dots, n$, and $C_{\mathcal{X}}(d) = C_{\mathcal{Y}}(d)$, for all degree d .*

Proof. Let $\mathcal{X} = \{P_1, \dots, P_m\}$ and $\mathcal{Y} = \{Q_1, \dots, Q_m\}$, where $P_i = (x_0 : \dots : x_n)$ and $Q_i = (a_0^{-1}x_0, \dots, a_n^{-1}x_n)$ for all $i = 0, \dots, n$. Let $v \in C_{\mathcal{X}}(d)$, then $v = (f(P_1) : \dots : f(P_m))$ for some $f \in S_d$. Define $g(X_0, \dots, X_n) = f(a_0X_0, \dots, a_nX_n) \in S_d$. It is easy to see that $v = (g(Q_1) : \dots : g(Q_m))$, so that $C_{\mathcal{X}}(d) \subset C_{\mathcal{Y}}(d)$. The proof of $C_{\mathcal{Y}}(d) \subset C_{\mathcal{X}}(d)$ is similar. \square

Thus we see that one may always assume that $1 \in A_j$, for all $j = 0, \dots, n$. We present now the special class of projective nested cartesian set for whose associated codes we will determine the minimum distance.

Definition 3.4. Let $K_0 \subset \cdots \subset K_n$ be subfields of K , with $|K_i| = d_i$ for all $0 \leq i \leq n$. Observe that $d_{i+1} = d_i^{r_i}$, for some $r_i \geq 1$ and $q = d_n^{r_n}$. Then $\mathcal{X} = [K_0 \times \cdots \times K_n]$ is a projective nested cartesian set which is called a *projective nested product of fields*.

Clearly \mathbb{P}^n is a projective nested product of fields, so our results on codes defined over such sets extend the results on projective Reed-Muller codes.

Definition 3.5. Let $g \in S$ a polynomial of degree d not necessarily homogeneous. We say that g is *homogeneous on \mathcal{X}* , and we write $g \in \tilde{S}_d$, if for every $i \in \{0, \dots, n\}$ and every $x = (0 : \cdots : 0 : 1 : x_{i+1} : \cdots : x_n) \in \mathcal{X}$ we have that for any given $c \in A_i^{\neq 0}$ there exists $\tilde{c} \in A_i^{\neq 0}$ such that

$$g(0, \dots, 0, c, cx_{i+1}, \dots, cx_n) = \tilde{c}g(0, \dots, 0, 1, x_{i+1}, \dots, x_n).$$

Definition 3.6. For a set $\mathcal{A} \subset \mathcal{X}$ and $f \in \tilde{S}_d \setminus I(\mathcal{A})$, define

$$Z_{\mathcal{A}}(f) := \{P \in \mathcal{A} \mid f(P) = 0\}.$$

In this way, for a codeword $v = (f(P_1), \dots, f(P_m)) \neq 0$, where $f(X) \in S_d \setminus I(\mathcal{X})_d$, the weight of v is $|\mathcal{X} \setminus Z_{\mathcal{X}}(f)|$, and the minimum distance of $C_{\mathcal{X}}(d)$ is

$$\delta_{\mathcal{X}}(d) = \min \{|\mathcal{X} \setminus Z_{\mathcal{X}}(f)| : f \in S_d \setminus I(\mathcal{X})_d\}.$$

Lemma 3.7. Let f be an element of \tilde{S}_d such that for all $t \leq j \leq n$ we have $Z_{\mathcal{X}}(X_j) \subset Z_{\mathcal{X}}(f)$. Then there exists $g_t(X)$ in $\tilde{S}_{d-(n-t+1)}$ such that $f - g_t \cdot X_t \cdots X_n \in I(\mathcal{X})$.

Proof. Write $f = g_n X_n + h_n$, where $h_n \in K[X_0, \dots, X_{n-1}]$. For any $P = (x_0 : \cdots : x_{n-1} : 0) \in \mathcal{X}$, we have $f(P) = 0$. This implies that $h_n \in I([K_0 \times \cdots \times K_{n-1}])$, and a fortiori we have $h_n \in I(\mathcal{X})$. By induction on α , suppose that for some $t+1 \leq \alpha \leq n$ we have $f = g_{\alpha} X_{\alpha} \cdots X_n + h_{\alpha}$, where $h_{\alpha} \in I(\mathcal{X})$. Write $g_{\alpha} = g_{\alpha-1} X_{\alpha-1} + \tilde{h}_{\alpha-1}$, where $\tilde{h}_{\alpha-1} \in K[X_0, \dots, X_{\alpha-2}, X_{\alpha}, \dots, X_n]$. For any $P = (x_0 : \cdots : x_{\alpha-2} : 0 : x_{\alpha} : \cdots : x_n) \in \mathcal{X}$, we have $f(P) = 0$. This implies $(\tilde{h}_{\alpha-1} X_{\alpha} \cdots X_n)(P) = 0$, which means $\tilde{h}_{\alpha-1} X_{\alpha} \cdots X_n \in I([K_0 \times \cdots \times K_{\alpha-2} \times K_{\alpha} \times \cdots \times K_n]) \subset I(\mathcal{X})$. We have then $f = g_{\alpha-1} X_{\alpha-1} \cdots X_n + \tilde{h}_{\alpha-1} X_{\alpha} \cdots X_n + h_{\alpha}$, where $\tilde{h}_{\alpha-1} X_{\alpha} \cdots X_n + h_{\alpha} \in I(\mathcal{X})$. By induction on α , our result is proved. It is easy to see that $g_t \in \tilde{S}_{d-(n-t+1)}$. \square

Proposition 3.8. Let \mathcal{X} be the projective nested product of fields over K_0, \dots, K_n , and let $f \notin I(\mathcal{X})$ be a not necessarily homogeneous polynomial on S of degree at most d and

homogeneous on \mathcal{X} . If $1 \leq d < \sum_{i=1}^n (d_i - 1)$, then

$$|\mathcal{X} \setminus Z_{\mathcal{X}}(f)| \geq (d_{k+1} - \ell) d_{k+2} \cdots d_n,$$

where $0 \leq k \leq n - 1$ and $0 \leq \ell < d_{k+1} - 1$ are the unique integers such that $d - 1 = \sum_{i=1}^k (d_i - 1) + \ell$.

Proof. We will make an induction on n . If $n = 1$, then $\mathcal{X} = [K_0 \times K_1]$ and let $x = (x_0 : x_1) \in \mathcal{X}$. Assume that $x_0 \neq 0$, since f is homogeneous on \mathcal{X} we have $f(x_0, x_1) = 0$ if and only if $f(1, x_1/x_0) = 0$. The last one is a polynomial of degree at most d on x_1/x_0 , which has no more than d roots. If f has a root on $(0 : 1)$, then writing $f = X_0 g + f_1$, with $f_1 \in K[X_1]$ we get that $f_1(a) = 0$ for all $a \in K_1$ hence $f(1, a) = 0$ if and only if $g(1, a) = 0$ (for all $a \in K_1$), and $g(1, X_1)$ has degree at most $d - 1$. Thus anyway we have

$$|\mathcal{X} \setminus Z_{\mathcal{X}}(f)| \geq (d_1 + 1) - d = d_1 - (d - 1).$$

So now we assume that the statement of the theorem holds for the product $[K_0 \times K_1 \times \cdots \times K_{n-1}]$. Define

$$\mathcal{Y}_n^* = [1 \times K_1 \times \cdots \times K_n] \text{ and } \mathcal{Y}_{n-1} = [0 \times K_1 \times \cdots \times K_n],$$

in particular $\mathcal{X} = \mathcal{Y}_n^* \cup \mathcal{Y}_{n-1}$. Let $f \notin I(\mathcal{X})$ be a polynomial of degree at most d , homogeneous on \mathcal{X} .

Suppose firstly that $f \in I(\mathcal{Y}_n^*)$ (so $f \notin I(\mathcal{Y}_{n-1})$). From Theorem 1.3 (and the fact that K_j is a finite field with d_j elements, for $j = 1, \dots, n$) we get that $I(\mathcal{Y}_n^*)$ is generated by $\tilde{\mathcal{G}} = \{X_j^{d_j} - X_j X_0^{d_j-1} \mid j = 1, \dots, n\}$. Endowing S with a graded-lexicographic order \prec such that $X_0 \prec X_1 \prec \cdots \prec X_n$ we get that $\text{lm}(X_j^{d_j} - X_j X_0^{d_j-1}) = X_j^{d_j}$, for all $j = 1, \dots, n$. Thus any pair of these leading monomials are coprime, so $\tilde{\mathcal{G}}$ is a Gröbner basis for $I(\mathcal{Y}_n^*)$, with respect to \prec (see [2, p. 104]). Dividing f by the elements of $\tilde{\mathcal{G}}$ we find polynomials g_j of degree at most $d - d_j$ ($j = 1, \dots, n$) such that $f(X) = \sum_{j=1}^n g_j(X)(X_j^{d_j} - X_j X_0^{d_j-1})$. Define $g(X) := \sum_{j=1}^n g_j(X)X_j$, which is a polynomial of degree $\tilde{d} \leq d - d_1 + 1$. Observe that $g|_{\mathcal{Y}_{n-1}} = f|_{\mathcal{Y}_{n-1}}$, which implies that for any $x = (0 : \cdots : 0 : 1 : x_{i+1} : \cdots : x_n)$ and any $c \in K_i^{\neq 0}$ there exists $\tilde{c} \in K_i^{\neq 0}$ such that $g(cx) = f(cx) = \tilde{c}f(x) = \tilde{c}g(x)$ so g is homogeneous on \mathcal{Y}_{n-1} . Since $f \notin I(\mathcal{Y}_{n-1})$, we must have $g \notin I(\mathcal{Y}_{n-1})$, and as

$\tilde{d} - 1 \leq d - 1 - (d_1 - 1) = \sum_{i=2}^k (d_i - 1) + \ell$, we can apply the induction hypothesis obtaining

$$|\mathcal{X} \setminus Z_{\mathcal{X}}(f)| = |\mathcal{Y}_{n-1} \setminus Z_{\mathcal{Y}_{n-1}}(g)| \geq (d_{k+1} - \ell) d_{k+2} \cdots d_n.$$

Suppose now that $f \in I(\mathcal{Y}_{n-1})$ and write $f = h + X_0 g$ where $h(X) = f(0, X_1, \dots, X_n)$. Since $f|_{\mathcal{Y}_{n-1}} = 0$ we have $h|_{\mathcal{Y}_{n-1}} = 0$ and a fortiori $h|_{\mathcal{Y}_n^*} = 0$ so $h \in I(\mathcal{X})$. Observe that $f|_{\mathcal{Y}_n^*} = g|_{\mathcal{Y}_n^*}$ and clearly the number of zeroes of g in \mathcal{Y}_n^* is the same of the number of zeroes of $g(1, X_1, \dots, X_n)$ in the cartesian product $K_1 \times \cdots \times K_n$. Since $\deg(g) \leq d - 1$ a lower bound for the number of nonzeros of g in \mathcal{Y}_n^* may be obtained from Theorem 1.2, and we have

$$|\mathcal{X} \setminus Z_{\mathcal{X}}(f)| = |\mathcal{Y}_n^* \setminus Z_{\mathcal{Y}_n^*}(g)| \geq (d_{k+1} - \ell) d_{k+2} \cdots d_n.$$

Finally suppose that $f \notin I(\mathcal{Y}_n^*)$ and $f \notin I(\mathcal{Y}_{n-1})$.

For $k = n - 1$, i.e. when $d = \sum_{i=1}^{n-1} (d_i - 1) + \ell + 1$, we have

$$|\mathcal{Y}_n^* \setminus Z_{\mathcal{Y}_n^*}(f)| \geq d_n - \ell - 1$$

since, as above, we may consider the number of nonzero points of $f(1, X_1, \dots, X_n)$ in $K_1 \times \cdots \times K_n$ and use Theorem 1.2. From $f \notin I(\mathcal{Y}_{n-1})$ we get

$$|\mathcal{Y}_{n-1} \setminus Z_{\mathcal{Y}_{n-1}}(f)| \geq 1,$$

which implies

$$|\mathcal{X} \setminus Z_{\mathcal{X}}(f)| \geq d_n - \ell$$

and settles the case $k = n - 1$. We treat now the case $k < n - 1$, and we start by assuming that $l + d_1 \leq d_{k+1}$.

We have that $d = \sum_{i=1}^k (d_i - 1) + \ell + 1$ and $d - 1 = \sum_{i=2}^k (d_i - 1) + \ell + d_1 - 1$, then

$$|\mathcal{Y}_n^* \setminus Z_{\mathcal{Y}_n^*}(f)| \geq (d_{k+1} - \ell - 1) d_{k+2} \cdots d_n,$$

$$|\mathcal{Y}_{n-1} \setminus Z_{\mathcal{Y}_{n-1}}(f)| \geq (d_{k+1} - (\ell + d_1 - 1)) d_{k+2} \cdots d_n \geq d_{k+2} \cdots d_n.$$

Adding both inequalities we obtain the desired result.

From now on we can assume that

$$f \notin I(\mathcal{Y}_n^*), f \notin I(\mathcal{Y}_{n-1}), 0 \leq k < n - 1 \text{ and } l + d_1 > d_{k+1}.$$

In particular $l \geq 1$. In what follows we generalize some methods used by Sørensen [10] to treat projective Reed-Muller codes. Define the set of hyperplanes

$$\Pi := \{\pi = Z(h) \subset \mathbb{P}^n \mid h = a_0 X_0 + \cdots + a_{n-1} X_{n-1} + X_n \in K_n[X]\}.$$

For all $\pi \in \Pi$, we want to estimate $|(\pi \cap \mathcal{X}) \setminus Z_{\mathcal{X}}(f)|$.

For each $h = a_0X_0 + \dots + a_{n-1}X_{n-1} + X_n$, define $H : \mathbb{P}^n \mapsto \mathbb{P}^n$ by

$$H(x_0, \dots, x_n) = (x_0 : \dots : x_{n-1} : h(x_0, \dots, x_n)).$$

It is easy to see that H is a projectivity that induces a bijection of \mathcal{X} and sends the plane π to the plane $Z(X_n)$, in fact

$$P \in \pi = Z(h) \iff H(P) \in Z(X_n).$$

It is also easy to check that $f(H(X)) := f(X_0, \dots, X_{n-1}, a_0X_0 + \dots + a_{n-1}X_{n-1} + X_n)$ is a polynomial of degree at most d and homogeneous on \mathcal{X} , and that the inverse projectivity H^{-1} is the one associated to $h^* = -a_0X_0 - \dots - a_{n-1}X_{n-1} + X_n$. Let $g_h(X) = f(H^{-1}(X))$, then we have a bijection between the zeroes of f in \mathcal{X} and the zeroes of g in $H(\mathcal{X}) (= \mathcal{X})$ given by

$$P \in Z_{\mathcal{X}}(f) \iff f(P) = 0 \iff g_h(H(P)) = 0 \iff H(P) \in Z_{\mathcal{X}}(g_h),$$

which implies that $H((Z(h) \cap \mathcal{X}) \setminus Z_{\mathcal{X}}(f)) = (Z(X_n) \cap \mathcal{X}) \setminus Z_{\mathcal{X}}(g_h)$.

To proceed we consider the following cases, regarding the possibility of $Z_{\mathcal{X}}(f)$ to contain or not a set $\pi \cap \mathcal{X}$, with $\pi \in \Pi$.

(a) Assume that $Z_{\mathcal{X}}(f)$ does not contain any set $\pi \cap \mathcal{X}$, where $\pi \in \Pi$, and define the set of pairs

$$A_f := \{(P, \pi) \in (\mathcal{X} \setminus Z_{\mathcal{X}}(f)) \times \Pi \mid P \in \pi\}.$$

Let $\mathcal{X}' = [K_0 \times \dots \times K_{n-1}]$ and for every $\pi = Z(h)$ let $g'_h(X_0, \dots, X_{n-1}) = g_h(X_0, \dots, X_{n-1}, 0)$. Since $Z(h) \cap \mathcal{X} \not\subset Z_{\mathcal{X}}(f)$ we have that g'_h does not vanish on \mathcal{X}' , is homogeneous on \mathcal{X}' and has degree at most d . Thus, from $|(Z(X_n) \cap \mathcal{X}) \setminus Z_{\mathcal{X}}(g_h)| = |\mathcal{X}' \setminus Z_{\mathcal{X}'}(g'_h)|$ and the induction hypothesis we get that

$$|Z(h) \cap \mathcal{X} \setminus Z_{\mathcal{X}}(f)| \geq (d_{k+1} - \ell) d_{k+2} \dots d_{n-1}.$$

So for each $\pi \in \Pi$ we have at least $(d_{k+1} - \ell) d_{k+2} \dots d_{n-1}$ points P such that $(P, \pi) \in A_f$. From $|\Pi| = d_n^n$ we have

$$(3.1) \quad |A_f| \geq (d_{k+1} - \ell) d_{k+2} \dots d_{n-1} d_n^n.$$

Let $P = (b_0 : \dots : b_n) \in \mathcal{X} \setminus Z_{\mathcal{X}}(f)$. If $(b_0 : \dots : b_{n-1}) \neq 0$ then there are d_n^{n-1} hyperplanes $\pi \in \Pi$ such that $P \in \pi$. If $P = (0 : \dots : 0 : 1)$, there is no hyperplane $\pi \in \Pi$ such that $P \in \pi$, so

$$(3.2) \quad |A_f| \leq |\mathcal{X} \setminus Z_{\mathcal{X}}(f)| d_n^{n-1}.$$

From (3.1) and (3.2) we get

$$|\mathcal{X} \setminus Z_{\mathcal{X}}(f)| \geq (d_{k+1} - \ell) d_{k+2} \cdots d_n.$$

(b) Assume that $Z_{\mathcal{X}}(f)$ contains a set $\pi \cap \mathcal{X}$, for some $\pi \in \Pi$. To complete the proof we will consider two subcases.

Subcase b.1: Assume that $d_{k+1} < d_n$. Applying the projectivity H corresponding to π and passing from $f(X)$ to $f(H^{-1}(X))$ we may assume that $\pi = Z(X_n)$. From Lemma 3.7 there exists a polynomial g of degree at most $d - 1$ and homogeneous on \mathcal{X} such that $f - gX_n \in I(\mathcal{X})$, which means $Z_{\mathcal{X}}(f) = Z_{\mathcal{X}}(gX_n)$. For $\tilde{\mathcal{X}} := [1 \times K_1 \times \cdots \times K_{n-1} \times K_n^{\neq 0}]$ we have $\mathcal{Y}_n^* \setminus Z_{\mathcal{Y}_n^*}(f) = \tilde{\mathcal{X}} \setminus Z_{\tilde{\mathcal{X}}}(g)$. As before we may get a lower bound for $\tilde{\mathcal{X}} \setminus Z_{\tilde{\mathcal{X}}}(g)$ by using Theorem 1.2 to obtain a lower bound for the number of nonzero points of $g(1, X_1, \dots, X_n)$ in $K_1 \times \cdots \times K_{n-1} \times K_n^{\neq 0} \in \mathbb{A}^n$. To do this we observe that $g(1, X_1, \dots, X_n)$ is a polynomial of degree at most $d - 1$, and also that $d_1 \leq \cdots \leq d_{n-1}$ and $d_{k+1} \leq d_n - 1$. Thus when we write $K_1, \dots, K_{n-1}, K_n^{\neq 0}$ in order of increasing size the set $K_n^{\neq 0}$ does not appear before K_{k+1} . In [8] the authors prove that this reordering does not affect the lower bound in Theorem 1.2 (2) so we get

$$|\tilde{\mathcal{X}} \setminus Z_{\tilde{\mathcal{X}}}(g)| \geq (d_{k+1} - \ell) d_{k+2} \cdots d_{n-1} (d_n - 1).$$

On the set \mathcal{Y}_{n-1} we can use the induction hypothesis, observing that $d - 1 = \sum_{i=2}^{k+1} (d_i - 1) +$

$\ell + d_1 - d_{k+1}$ and $0 < \ell + d_1 - d_{k+1} \leq d_{k+2} - 1$, so

$$(3.3) \quad |\mathcal{Y}_{n-1} \setminus Z_{\mathcal{Y}_{n-1}}(f)| \geq (d_{k+2} - (\ell + d_1 - d_{k+1})) d_{k+3} \cdots d_n \geq (d_{k+1} - \ell) d_{k+2} \cdots d_{n-1}.$$

Adding both inequalities, we obtain the desired result.

Subcase b.2: Assume that $d_{k+1} = d_n$. Let $t \in \{1, \dots, k+1\}$ be the least index such that $K_t = K_{t+1} = \cdots = K_n$. For $t \leq j \leq n$ let

$$\Pi_j = \{\pi = Z(h) \subset \mathbb{P}^n \mid h = a_0X_0 + \cdots + a_{j-1}X_{j-1} + X_j + a_{j+1}X_{j+1} + \cdots + a_nX_n \in K_n[X]\}.$$

If for some $j \in \{t, \dots, n\}$ all sets $\pi \cap \mathcal{X}$, with $\pi \in \Pi_j$, are not contained in $Z_{\mathcal{X}}(f)$ then we may use an argument similar to the one used in (a) above to obtain the desired result.

In this argument we will use Π_j instead of Π , $\mathcal{X}' = [K_0 \times \cdots \times \widehat{K_j} \times \cdots \times K_n]$ instead of \mathcal{X}' (where $K_0 \times \cdots \times \widehat{K_j} \times \cdots \times K_n$ means that we omit the set K_j in the product) and for every $h = a_0X_0 + \cdots + a_{j-1}X_{j-1} + X_j + a_{j+1}X_{j+1} + \cdots + a_nX_n \in K_n[X]$ we will set $g'_h(X_0, \dots, \widehat{X_j}, \dots, X_n) = f(X_0, \dots, X_{j-1}, -a_0X_0 - \cdots - a_{j-1}X_{j-1} - a_{j+1}X_{j+1} - \cdots -$

$a_n X_n, X_{j+1}, \dots, X_n$); at the end we use that $|\Pi_j| = d_n^n = d_j^n$ to conclude the argument and prove the result.

If for all $t \leq j \leq n$ there exists $Z(h_j) = \pi_j \in \Pi_j$ such that $\pi_j \cap \mathcal{X} \subset Z_{\mathcal{X}}(f)$ then let H be the projectivity defined by

$$H(x_0, \dots, x_n) = (x_0 : \dots : x_{t-1} : h_t(x_0, \dots, x_n) : x_{t+1} : \dots : x_n).$$

As before, passing from $f(X)$ to $f(H^{-1}(X))$ we may assume that $Z(X_t) \cap \mathcal{X} \subset Z_{\mathcal{X}}(f)$. If all sets $\pi \cap \mathcal{X}$, with $\pi \in \Pi_{t+1}$, are not contained in $Z_{\mathcal{X}}(f)$ then again we may use an argument similar to the one used in (a) above to get the result. If there is some $\pi \in \Pi_{t+1}$ such that $\pi \cap \mathcal{X} \subset Z_{\mathcal{X}}(f)$ then using an appropriate projectivity we may assume that $Z(X_{t+1}) \cap \mathcal{X} \subset Z_{\mathcal{X}}(f)$ (note that $Z(X_t) \cap \mathcal{X} \subset Z_{\mathcal{X}}(f)$ continues to hold). Proceeding in this manner, we either get the result or we get that $Z(X_j) \cap \mathcal{X} \subset Z_{\mathcal{X}}(f)$ for all $j = t, \dots, n$, which we assume from now on. From Lemma 3.7, there exists a polynomial $g(X)$ of degree at most $d - (n - t + 1)$, homogeneous on \mathcal{X} , such that $f = g \cdot X_t \cdots X_n$. From $f \notin I(\mathcal{Y}_n^*)$ we get that g is not zero on the set $\mathcal{A} = [1 \times K_1 \times \cdots \times K_t^* \times \cdots \times K_n^*]$ and also that $|\mathcal{Y}_n^* \setminus Z_{\mathcal{Y}_n^*}(f)| = |\mathcal{A} \setminus Z_{\mathcal{A}}(g)|$. The number of nonzero points of g in \mathcal{A} is the same of the number of nonzero points of $g(1, X_1, \dots, X_n)$ in $K_1 \times \cdots \times K_t^* \times \cdots \times K_n^* \in \mathbb{A}^n$. Observe that from the definition of t we get $d_1 \leq \cdots \leq d_{t-1} \leq d_t - 1 = \cdots = d_n - 1$ so we may apply Theorem 1.2, noting that $\deg(1, X_1, \dots, X_n) \leq d - 1 - (n - t)$. To apply that result we write

$$(3.4) \quad d - 1 - (n - t) = \sum_{i=1}^{t-1} (d_i - 1) + \sum_{i=t}^k ((d_i - 1) - 1) + \ell - (n - k - 1) = \sum_{i=1}^{\alpha} (\tilde{d}_i - 1) + \tilde{\ell},$$

where \tilde{d}_i , $0 \leq \alpha \leq k$ and $\tilde{\ell}$ are defined by

$$\tilde{d}_i = \begin{cases} d_i & \text{if } 1 \leq i < t, \\ d_i - 1 & \text{if } t \leq i \leq n, \end{cases}$$

$$0 \leq \tilde{\ell} = \sum_{i=\alpha+1}^k (\tilde{d}_i - 1) + \ell - (n - k - 1) < \tilde{d}_{\alpha+1} - 1$$

(we note that if $t = k + 1$ then we omit the term $\sum_{i=t}^k ((d_i - 1) - 1)$ in (3.4)). With this notation, from Theorem 1.2 we have

$$|\mathcal{A} \setminus Z_{\mathcal{A}}(g)| \geq (\tilde{d}_{\alpha+1} - \tilde{\ell}) \tilde{d}_{\alpha+2} \cdots \tilde{d}_n.$$

Let $a_{\alpha+1} = d_{\alpha+1} - \tilde{d}_{\alpha+1} + \tilde{\ell}$ and $a_j = d_j - \tilde{d}_j$ for $j = \alpha + 2, \dots, n - 1$, then

$$(\tilde{d}_{\alpha+1} - \tilde{\ell})\tilde{d}_{\alpha+2} \cdots \tilde{d}_{n-1} = \prod_{i=\alpha+1}^{n-1} (d_i - a_i),$$

and we have

$$\begin{aligned} \sum_{i=\alpha+1}^{n-1} a_i &= (d_{\alpha+1} - \tilde{d}_{\alpha+1} + \tilde{\ell}) + \sum_{i=\alpha+2}^{n-1} (d_i - \tilde{d}_i) = \tilde{\ell} + \sum_{i=\alpha+1}^{n-1} (d_i - \tilde{d}_i) \\ &= \sum_{i=\alpha+1}^k (\tilde{d}_i - 1) + \ell - (n - k - 1) + \sum_{i=\alpha+1}^k (d_i - \tilde{d}_i) + (n - 1 - k) \\ &= \sum_{i=\alpha+1}^k (d_i - 1) + \ell. \end{aligned}$$

Thus, from Lemma 1.4 we get $\prod_{i=\alpha+1}^{n-1} (d_i - a_i) \geq (d_{k+1} - \ell)d_{k+2} \cdots d_{n-1}$, and a fortiori

$$|\mathcal{A} \setminus Z_{\mathcal{A}}(g)| \geq (d_{k+1} - \ell)d_{k+2} \cdots d_{n-1}(d_n - 1).$$

From the induction hypothesis, and similarly as (3.3), we have

$$|\mathcal{Y}_{n-1} \setminus Z_{\mathcal{Y}_{n-1}}(f)| \geq (d_{k+1} - \ell)d_{k+2} \cdots d_{n-1}.$$

and adding both inequalities we obtain the desired result, which concludes the proof of the Proposition. \square

We come to the main result of this section.

Theorem 3.9. *If \mathcal{X} is the projective nested product of fields over K_0, \dots, K_n , then the minimum distance of $C_{\mathcal{X}}(d)$ is given by*

$$\delta_{\mathcal{X}}(d) = \begin{cases} (d_{k+1} - \ell) d_{k+2} \cdots d_n & \text{if } 1 \leq d \leq \sum_{i=1}^n (d_i - 1), \\ 1 & \text{if } \sum_{i=1}^n (d_i - 1) < d, \end{cases}$$

where $0 \leq k \leq n - 1$ and $0 \leq \ell < d_{k+1} - 1$ are the unique integers such that

$$d - 1 = \sum_{i=1}^k (d_i - 1) + \ell.$$

Proof. Now it is immediate by Proposition 3.8 and Lemma 3.1. \square

As a consequence of our main results we recover the formula for the parameters of Projective Reed-Muller codes.

Corollary 3.10. ([10, Theorem 1]; [9, Proposition 12]) *The Projective Reed-Muller code $PC_d(n, q)$ is an $[|\mathbb{P}^n|, \dim C_{\mathbb{P}^n}(d), \delta_{\mathbb{P}^n}(d)]$ -code where*

- (a) $|\mathbb{P}^n| = (q^{n+1} - 1)/(q - 1),$
- (b) $\dim C_{\mathbb{P}^n}(d) = \sum_{j=0}^n \sum_{k=0}^j (-1)^k \binom{j}{k} \binom{j+d-1-kq}{d-1-kq} \text{ and}$
- (c)

$$\delta_{\mathbb{P}^n}(d) = \begin{cases} q^n & \text{if } 1 = d, \\ (q - \ell) q^{n-k-1} & \text{if } 1 < d \leq n(q - 1), \\ 1 & \text{if } n(q - 1) < d; \end{cases}$$

here $0 \leq k \leq n - 1$ and $1 \leq \ell \leq d_{k+1} - 1$ are the unique integers such that $d = 1 + k(q - 1) + \ell$.

Proof. Using Remark 2.3 and Theorems 2.9 and 3.9 we have the result. \square

Now we present a relationship between the parameters of codes defined over a projective nested product of fields and certain affine cartesian codes.

Corollary 3.11. *Let K_0, \dots, K_n be subfields of K such that $\mathcal{X} = [K_0 \times K_1 \times \dots \times K_n]$ is a projective nested product of fields and let $\mathcal{X}_i^* = K_{n+1-i} \times \dots \times K_n \subset \mathbb{A}^i$, where $i = 1 \dots, n$. Set $\mathcal{X}_0^* = \{1\}$ If*

$$C_{\mathcal{X}}(d) \text{ is a } [|\mathcal{X}|, \dim C_{\mathcal{X}}(d), \delta_{\mathcal{X}}(d)] \text{-code}$$

and

$$C_{\mathcal{X}_i^*}(d) \text{ is a } [|\mathcal{X}_i^*|, \dim C_{\mathcal{X}_i^*}(d), \delta_{\mathcal{X}_i^*}(d)] \text{-code,}$$

then

$$|\mathcal{X}| = \sum_{i=0}^n |\mathcal{X}_i^*|, \quad \dim C_{\mathcal{X}}(d) = \sum_{i=0}^n \dim C_{\mathcal{X}_i^*}(d - 1) \quad \text{and} \quad \delta_{\mathcal{X}}(d) = \delta_{\mathcal{X}_n^*}(d - 1),$$

where $\mathcal{X}_0^* = [1]$ and $\delta_{\mathcal{X}_n^*}(0) := d_1 \dots d_n$.

Proof. It is a consequence of Theorems 2.9 and 3.9 and [8, Corollary 3.8]. \square

Example 3.12. Let $K = \mathbb{F}_{25}$ be a finite field with 25 elements and let $K_0 = K_1 = \mathbb{F}_5, K_2 = \mathbb{F}_{25}$ be subsets of K . Then $\mathcal{X} = [K_0 \times K_1 \times K_2]$ is a projective nested cartesian

product, and the length, the dimension and the minimum distance of the code $C_{\mathcal{X}}(d)$ are:

| d | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 25 |
|---------------------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $ \mathcal{X} $ | 151 | 151 | 151 | 151 | 151 | 151 | 151 | 151 | 151 | 151 | 151 |
| $\dim C_{\mathcal{X}}(d)$ | 3 | 6 | 10 | 15 | 21 | 27 | 33 | 39 | 45 | 51 | 141 |
| $\delta_{\mathcal{X}}(d)$ | 125 | 100 | 75 | 50 | 25 | 24 | 23 | 22 | 21 | 20 | 5 |

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FACULDADE DE MATEMÁTICA, UNIVERSIDADE FEDERAL DE UBERLÂNDIA, AV. J. N. ÁVILA 2121, 38.408-902 - UBERLÂNDIA - MG, BRAZIL

E-mail address: cicero@ufu.br

FACULDADE DE MATEMÁTICA, UNIVERSIDADE FEDERAL DE UBERLÂNDIA, AV. J. N. ÁVILA 2121, 38.408-902 - UBERLÂNDIA - MG, BRAZIL

E-mail address: gonzalo@famat.ufu.br

DEPARTAMENTO DE MATEMÁTICAS, CENTRO DE INVESTIGACIÓN Y DE ESTUDIOS AVANZADOS DEL IPN, APARTADO POSTAL 14-740, 07000 MEXICO CITY, D.F.

E-mail address: hlopez@math.cinvestav.mx